TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 3, March 1995

ON CAPPELL-SHANESON'S HOMOLOGY L-CLASSES OF SINGULAR ALGEBRAIC VARIETIES

SHOJI YOKURA

ABSTRACT. S. Cappell and J. Shaneson (*Stratifiable maps and topological invariants*, J. Amer. Math. Soc. **4** (1991), 521–551) have recently developed a theory of homology L-classes, extending Goresky-MacPherson's homology L-classes. In this paper we show that Cappell-Shaneson's homology L-classes for compact complex, possibly singular, algebraic varieties can be interpreted as a unique natural transformation from a covariant cobordism function Ω to the **Q**-homology functor $H_*(\ ; \mathbf{Q})$ satisfying a certain normalization condition, just like MacPherson's Chern classes and Baum-Fulton-MacPherson's Todd classes.

INTRODUCTION

In this paper the category on which to work is that of compact complex, possibly singular, algebraic varieties.

Cappell and Shaneson [CS1] have recently developed a theory of homology L-classes, extending Goresky-MacPherson's homology L-classes [GM1], by using the sheaf-theoretic methods of [Bo, GM2] and some of the foundational topological aspects of the theory of perverse sheaves [BBD, §§1, 2]. The aim of this paper is to give another formulation to Cappell-Shaneson's homology L-classes of compact complex, possibly singular, algebraic varieties, strengthening their results. This work is motivated by having noticed one of their results [CS1, (5.5) Proposition] and having read their recent announcement [CS2].

Our results are the following

Theorem. (i) *Let $\Omega(X)$ be the set of all the cobordism classes of self-dual complexes of sheaves on a compact complex algebraic variety X. Then $\Omega(X)$ becomes an Abelian group.*

(ii) *Let $\mathbf{V}$ be the category of compact complex algebraic varieties and* **Ab** *the category of Abelian groups. Then the correspondence $\Omega: \mathbf{V} \to \mathbf{Ab}$ assigning the cobordism group $\Omega(X)$ of self-dual complexes of sheaves to each object $X \in$ Obj($\mathbf{V}$) becomes a covariant functor with the pushforward $f_\# := \Omega(f): \Omega(X) \to \Omega(Y)$, for a morphism $f: X \to Y$, defined by*

$$f_\#([S^\cdot]) := [Rf_*S^\cdot[-\mathrm{reldim}(f)]],$$

Received by the editors November 1, 1993; originally communicated to the *Proceedings of the AMS* by Eric Friedlander.

1991 *Mathematics Subject Classification*. Primary 14B05, 14C40, 14F99, 18E30, 18F99, 57R20.

Partially supported by Grant-in-Aid for Scientific Research (No. 06640162), the Ministry of Education, Science, and Culture.

where Rf_* *is the right derived functor and* $\operatorname{reldim}(f) := \dim X - \dim Y$ *is the relative dimension of the morphism* f.

(iii) (*Cappell-Shaneson's homology L-class*). *There exists a unique natural transformation*

$$L_*: \Omega(\ ) \to H_*(\ ; \mathbf{Q})$$

satisfying the extra condition ("smooth L-condition") that if X *is smooth, then*

$$L_*(\mathbf{Q}_X[2\dim X]) = L^*(X) \cap [X],$$

where $\mathbf{Q}_X$ *is the constant sheaf (considered as a complex concentrated in degree* 0), $\dim X$ *is the complex dimension of* X, *and* $L^*(X)$ *is the usual total Hirzebruch's cohomology L-class of the tangent bundle* T_X.

For the cobordism $[S^\cdot]$ of a self-dual complex $S^\cdot$ of sheaves $L_*([S^\cdot])$ is the Cappell-Shaneson's homology L-class and for the cobordism class $[IC^\cdot(X; \mathbf{Q})]$ of the intersection sheaf complex $IC^\cdot(X; \mathbf{Q})$ with the rational coefficients [GM2] $L_*([IC^\cdot(X; \mathbf{Q})])$ is the Goresky-MacPherson's homology L-class [GM1]. The statement of the theorem is similar to that of the theorem on Chern-MacPherson classes [Mac] and that of the theorem on Baum-Fulton-MacPherson's Todd classes [BFM]. Of course these two well-known theorems are principal motivations for the above theorem. The basic work is done by Cappell and Shaneson in [CS1], and what we want to do in this paper is to show that we can prove the above theorem by further study of their work. We will notice that the relationship between the Goresky-MacPherson homology L-class and the Cappell-Shaneson homology L-class is just like the relationship between the Chern-Mather class and the Chern-MacPherson class. The reference for the definitions given in this paper is [CS1].

1. Covariant cobordism functor Ω

Definition (1.1) (self-dual complexes of sheaves). Let $S^\cdot$ be a bounded constructible complex of sheaves on X. If we have the following isomorphism in the derived category of constructible complexes of sheaves

$$S^\cdot \cong \mathbf{D}(S^\cdot)[2\dim X],$$

where $\dim X$ is the complex dimension of X and $\mathbf{D}$ is the Borel-Moore-Verdier dualizing functor [Bo], then $S^\cdot$ is called *self-dual.*

Remark (1.2). We note that $\mathbf{D}(S^\cdot[k]) \cong \mathbf{D}(S^\cdot)[-k]$, which follows from the isomorphism

$$\mathbf{D}(S^\cdot) \cong \operatorname{RHom}^\cdot(S^\cdot, D^\cdot_X),$$

where $D^\cdot_X$ is the Verdier dualizing complex. Hence the above self-duality formula is

$$S^\cdot \cong \mathbf{D}(S^\cdot[-2\dim X]).$$

Definition (1.3) (elementary cobordism). Let $X^\cdot \xrightarrow{u} S^\cdot \xrightarrow{v} Z^\cdot$ be morphisms in the derived category with $v \circ u = 0$, and furthermore suppose that there exists an isomorphism

$$Z^\cdot \cong \mathbf{D}(X^\cdot)[2\dim X]$$

such that the following diagram commutes

$$\begin{array}{ccc} S^\cdot & \xrightarrow{v} & Z^\cdot \\ \cong\downarrow & & \cong\downarrow \\ \mathbf{D}(S^\cdot)[2\dim X] & \xrightarrow{D(u)[2\dim X]} & \mathbf{D}(X^\cdot)[2\dim X] \end{array}$$

Then the algebraic mapping cone (see [CS1, §2])

$$S_1^\cdot := C^\cdot_{v,u} = X^\cdot[1] \oplus S^\cdot \oplus Z^\cdot[-1]$$

is also a self-dual complex of sheaves on X. Then we say that $S_1^\cdot$ is obtained from $S^\cdot$ by an elementary cobordism or $S_1^\cdot$ is *elementarily cobordant* to $S^\cdot$.

Definition (1.4) (cobordism). We say that $S^\cdot$ is cobordant to $S^{\cdot\#}$ if there is a finite sequence $S^\cdot = A_0^\cdot, A_1^\cdot, \dots, A_n^\cdot = S^{\cdot\#}$ such that each $A_i^\cdot$ is elementarily cobordant to $A_{i-1}^\cdot$.

Note that the cobordism is an equivalence relation.

Lemma (1.5). *Let $\Omega(X)$ be the set of all the cobordism classes of self-dual complexes of sheaves on a compact complex algebraic variety X. Then $\Omega(X)$ becomes an abelian group, which shall be called the cobordism group of self-dual complexes.*

Proof. The group operation (addition operation) is defined by

$$[S_1^\cdot] + [S_2^\cdot] := [S_1^\cdot \oplus S_2^\cdot].$$

Since $[S_1^\cdot]+[S_2^\cdot] = [S_1^\cdot \oplus S_2^\cdot] = [S_2^\cdot \oplus S_1^\cdot] = [S_2^\cdot]+[S_1^\cdot]$, the operation is commutative. The cobordism class $0 := [0^\cdot]$ of the trivial complex $0^\cdot$ is the identity element of $\Omega(X)$. It follows from [CS1, Proof of (2.3) Proposition] that $S^\cdot \oplus -S^\cdot$ is cobordant to $0^\cdot$, hence $[S^\cdot]+[-S^\cdot] = [S^\cdot \oplus -S^\cdot] = [0^\cdot] = 0$, where $-S^\cdot$ is the complex $S^\cdot$ with the negative $-d$ of the isomorphism $d\colon S^\cdot \cong \mathbf{D}(S^\cdot)[2\dim X]$. Thus $-[S^\cdot] = [-S^\cdot]$. Therefore $\Omega(X)$ becomes an abelian group.

Proposition (1.6). *Let* $\mathbf{V}$ *be the category of compact algebraic varieties and* $\mathbf{Ab}$ *be the category of abelian groups. The correspondence* $\Omega\colon \mathbf{V} \to \mathbf{Ab}$ *assigning the cobordism group $\Omega(X)$ of self-dual complexes of sheaves to each object $X \in$* $\mathrm{Obj}(\mathbf{V})$ *becomes a covariant functor with the pushforward $f_\# := \Omega(f)\colon \Omega(X) \to \Omega(Y)$, for a morphism $f\colon X \to Y$, defined by*

$$f_\#([S^\cdot]) := [Rf_* S^\cdot[-\mathrm{reldim}(f)]],$$

where Rf_* is the right-derived functor and $\mathrm{reldim}(f) := \dim X - \dim Y$ *is the relative dimension of the morphism f. Namely, for morphisms $g\colon X \to Y$ and $f\colon Y \to Z$, $(f \circ g)_\# = f_\# \circ g_\#$.*

Proof. It follows from [GM2, Proposition in §1.9] that $Rf_* S^\cdot[-\mathrm{reldim}(f)]$ is also a constructible complex of sheaves on Y. Also we note the following well-known identity concerning the Verdier dual: (note $f_! = f_*$ since f is proper)

(1.6.1) $\mathbf{D}(Rf_* A^\cdot) \cong Rf_*(\mathbf{D}A^\cdot)$ (e.g., see [GM2, §1.13]).

Then the proof of the self-duality of $Rf_*S^\bullet[-\operatorname{reldim}(f)]$ goes as follows:

$$\begin{aligned}
Rf_*S^\bullet[-\operatorname{reldim}(f)] &= Rf_*S^\bullet[-(\dim X - \dim Y)] \\
&= Rf_*S^\bullet[\dim X - \dim Y + 2\dim Y - 2\dim X] \\
&\cong Rf_*\mathbf{D}(S^\bullet)[\dim X - \dim Y + 2\dim Y] \\
&\qquad \text{(because } S^\bullet \cong \mathbf{D}(S^\bullet)[2\dim X]) \\
&\cong \mathbf{D}(Rf_*S^\bullet)[\dim X - \dim Y + 2\dim Y] \quad \text{(by (1.6.1))} \\
&\cong \mathbf{D}(Rf_*S^\bullet)[\dim X - \dim Y][2\dim Y] \\
&\cong \mathbf{D}(Rf_*S^\bullet[-(\dim X - \dim Y)])[2\dim Y] \quad \text{(by Remark (1.2))} \\
&= \mathbf{D}(Rf_*S^\bullet[-\operatorname{reldim}(f)])[2\dim Y].
\end{aligned}$$

Now we want to make the following

Claim. If $S_1^\bullet$ *and* $S_2^\bullet$ *are cobordant, then*

$$Rf_*S_1^\bullet[-\operatorname{reldim}(f)] \quad \textit{and} \quad Rf_*S_2^\bullet[-\operatorname{reldim}(f)]$$

are also cobordant.

This claim implies that the above definition of

$$f_\#([S^\bullet]) := [Rf_*S^\bullet[-\operatorname{reldim}(f)]]$$

is well defined. We prove this claim later and show the functoriality of $f_\#$, i.e.,

$$f_\#(g_\#[S^\bullet]) = (f\circ g)_\#([S^\bullet]).$$

Indeed,

$$\begin{aligned}
f_\#(g_\#[S^\bullet]) &= f_\#([Rg_*S^\bullet[-\operatorname{reldim}(g)]]) \\
&= [Rf_*(Rg_*S^\bullet[-\operatorname{reldim}(g)])[-\operatorname{reldim}(f)]] \\
&= [Rf_*\circ Rg_*S^\bullet[-\operatorname{reldim}(g)][-\operatorname{reldim}(f)]] \\
&= [R(f\circ g)_*S^\bullet[-\operatorname{reldim}(g) - \operatorname{reldim}(f)]] \\
&= [R(f\circ g)_*S^\bullet[-\operatorname{reldim}(f\circ g)]] \\
&= (f\circ g)_\#([S^\bullet]).
\end{aligned}$$

Now it remains to prove the above claim.

Proof of Claim. Since cobordism is a finite succession of elementary cobordisms, it suffices to show that if $S_1^\bullet$ and $S_2^\bullet$ are elementarily cobordant, then $Rf_*S_1^\bullet[-\operatorname{reldim}(f)]$ and $Rf_*S_2^\bullet[-\operatorname{reldim}(f)]$ are also elementarily cobordant. For the sake of simplicity, we denote the pushforward operation $Rf_*S^\bullet[-\operatorname{reldim}(f)]$ simply by $Rf_{[*]}S^\bullet$. To prove the claim, here we recall the definition of the elementary cobordism between $S_1^\bullet$ and $S_2^\bullet$. Let $X^\bullet \xrightarrow{u} S_1^\bullet \xrightarrow{v} Z^\bullet$ be morphisms in the derived category with $v\cdot u = 0$, and furthermore suppose that there exists an isomorphism

$$Z^\bullet \cong \mathbf{D}(X^\bullet)[2\dim X]$$

such that the following diagram commutes

$$\begin{array}{ccc}
S_1^\bullet & \xrightarrow{v} & Z^\bullet \\
\cong\downarrow & & \cong\downarrow \\
\mathbf{D}(S_1^\bullet)[2\dim X] & \xrightarrow{D(u)[2\dim X]} & \mathbf{D}(X^\bullet)[2\dim X].
\end{array} \qquad (1.6.2)$$

Then $S_2^\cdot = C_{v,u}^\cdot$ is the mapping cone and is by definition elementarily cobordant to $S_1^\cdot$. By applying the operation $Rf_{[*]}$ we get morphisms $Rf_{[*]}X^\cdot \to Rf_{[*]}S_1^\cdot \to Rf_{[*]}Z^\cdot$ in the derived category with $Rf_{[*]}v \circ Rf_{[*]}u = Rf_{[*]}(v \circ u) = 0$. Hence we get the mapping cone $C^\cdot_{Rf_{[*]}v, Rf_{[*]}u}$. Here we note that $Rf_{[*]}S_2^\cdot = Rf_{[*]}C_{v,u}^\cdot = C^\cdot_{Rf_{[*]}v, Rf_{[*]}u}$. So if we can show the following commutative diagram, then we are done.

$$\begin{array}{ccc} Rf_{[*]}S_1^\cdot & \xrightarrow{Rf_{[*]}v} & Rf_{[*]}Z^\cdot \\ \cong\downarrow & & \cong\downarrow \\ \mathbf{D}(Rf_{[*]}S_1^\cdot)[2\dim Y] & \xrightarrow{\mathbf{D}(Rf_{[*]}u)[2\dim Y]} & \mathbf{D}(Rf_{[*]}X^\cdot)[2\dim Y] \end{array} \tag{1.6.3}$$

Proof of (1.6.3). Let $\dim X = n$ and $\dim Y = m$

$$\begin{array}{ccccl} Rf_{[*]}S_1^\cdot & \to & Rf_{[*]}Z^\cdot & & \\ \| & & \| & & \\ Rf_*S_1^\cdot[m-n] & \to & Rf_*Z^\cdot[m-n] & & \\ \cong\downarrow & & \cong\downarrow & & \text{(by applying } Rf_{[*]} \\ & & & & \text{on (1.6.2))} \\ Rf_*(\mathbf{D}(S_1^\cdot)[2n])[m-n] & \to & Rf_*(\mathbf{D}(X^\cdot)[2n])[m-n] & & \\ \cong\downarrow & & \cong\downarrow & & \\ Rf_*(\mathbf{D}(S_1^\cdot))[m+n] & \to & Rf_*(\mathbf{D}(X^\cdot))[m+n] & & \\ \cong\downarrow & & \cong\downarrow & & \\ Rf_*(\mathbf{D}(S_1^\cdot[m-n]))[2m] & \to & Rf_*(\mathbf{D}(X^\cdot[m-n]))[2m] & & \\ \cong\downarrow & & \cong\downarrow & & \text{(by (1.6.1))} \\ \mathbf{D}(Rf_*S_1^\cdot[m-n])[2m] & \to & \mathbf{D}(Rf_*X^\cdot[m-n])[2m] & & \\ \| & & \| & & \\ \mathbf{D}(Rf_{[*]}S_1^\cdot)[2\dim Y] & \to & \mathbf{D}(Rf_{[*]}X^\cdot)[2\dim Y] & & \end{array}$$

Thus, by the definition of the elementary cobordism, $Rf_{[*]}S_2^\cdot = Rf_{[*]}C_{v,u}^\cdot \cong C^\cdot_{Rf_{[*]}v, Rf_{[*]}u}$ is elementarily cobordant to $Rf_{[*]}S_1^\cdot$. □

2. Cappell-Shaneson's homology L-class

Cappell-Shaneson's homology L-class ([CS1, §5. Characteristic classes] satisfies the following properties:

(CSL-1) [CS1, (5.1) Theorem]. For $S^\cdot$ a self-dual complex of sheaves on X,

$$L_*(S^\cdot) \in H_*(X; \mathbf{Q}).$$

(CSL-2) [CS1, (5.2) Proposition]. If $S_1^\cdot$ and $S_2^\cdot$ are cobordant, then

$$L_*(S_1^\cdot) = L_*(S_2^\cdot).$$

(CSL-3) [CS1, (5.4) Proposition]. If $S_1^\cdot$ and $S_2^\cdot$ are self-dual complexes of sheaves on X, then

$$L_*(S_1^\cdot \oplus S_2^\cdot) = L_*(S_1^\cdot) + L_*(S_2^\cdot).$$

These three properties imply the following

Corollary (2.1). *Cappell-Shaneson's homology L-class* L_* *is a transformation from the covariant cobordism functor* Ω *of self-dual complexes to the* **Q**-*homology functor* $H_*(\ ;\mathbf{Q})$.

Furthermore, Cappell and Shaneson prove the following

Theorem (2.2) ([CS1, (5.5) Proposition]). *Let* $f\colon X^p \to Y^m$ *be a stratified map of compact oriented Whitney stratified pseudomanifolds. (Here* p *and* m *are real dimensions.) Let* S^* *be a self-dual complex of sheaves on* X. *Assume that* $p-m$ *is even, and let* $t=(p-m)/2$. *Then*

$$L_*(Rf_*S^\bullet[-t]) = f_*L_*(S^\bullet).$$

As a corollary of this theorem we have

Corollary (2.3). *Cappell-Shaneson's homology L-class* L_* *is actually a natural transformation from the covariant cobordism functor* Ω *of self-dual complexes to the* **Q**-*homology functor* $H_*(;\mathbf{Q})$.

It is Theorem (2.2), i.e., [CS1, (5.5) Proposition], that has led us to this present work.

There are of course infinitely many natural transformations from the functor Ω to the **Q**-homology functor $H_*(\ ;\mathbf{Q})$. However, we can claim the following uniqueness theorem of Cappell-Shaneson's homology L-class $L_*\colon \Omega(\ ) \to H_*(\ ;\mathbf{Q})$, which is different from the uniqueness theorem of Cappell-Shaneson [CS1, (5.1) Theorem].

Theorem (2.4). *Cappell-Shaneson's homology L-class* L_* *is the unique natural transformation*

$$L_*\colon \Omega(\ ) \to H_*(\ ;\mathbf{Q})$$

satisfying the extra condition ("smooth L-condition") that if X *is smooth, then*

$$L_*([\mathbf{Q}_X[2\dim X]]) = L^*(X)\cap[X],$$

where $L^*(X)$ *is the usual Hirzebruch cohomology L-class of* X.

Proof. Cappell-Shaneson's homology L-class L_* also satisfies (see [CS1, §5])

(CSL-4) $L_*(IC^\bullet(X;\mathbf{Q})) = L^{\mathrm{GM}}(X)$,

the Goresky-MacPherson homology L-class [GM1].

Since if X is smooth $IC^\bullet(X;\mathbf{Q}) = \mathbf{Q}_X[2\dim X]$ and $L^{\mathrm{GM}}(X) = L^*(X)\cap[X]$, it is obvious that Cappell-Shaneson's homology L-class L_* satisfies the above "smooth L-condition".

Uniqueness. We prove that the uniqueness of L_* follows from the "smooth L-condition" and the resolution of singularities. First we show the following

Lemma (2.4.1) (cf. [CS1, (4.7) Theorem]). *For any self-dual complex* $S^\bullet$ *of sheaves on* X, *there exist finitely many subvarieties* W*'s of* X *and integers* n_W*'s such that*

$$[S^\bullet] = \sum n_W j_{W\#}[IC^\bullet(W;\mathbf{Q})]$$

where $j_W\colon W\to X$ *is the inclusion map.*

Proof. It is done by induction in the dimension of the support. Let $S^\bullet$ be a self-dual complex of sheaves on a point $\{x\}$; then by [CS1, (4.7) Theorem]

$$S^\bullet = n_{\{x\}}IC^\bullet(\{x\};\mathbf{Q}) \quad \text{(up to cobordism)},$$

where $n_{\{x\}} = \operatorname{rank} H^0(\{x\}: S^\cdot)$. Hence, if $\dim X = 0$, then for any self-dual complex of sheaves on X, the statement of the above lemma holds.

Suppose that the statement is true for $\dim X < k$, and prove the statement for $\dim X = k$. Let $S^\cdot$ be a self-dual complex of sheaves on X. Then, by [CS1, (4.7) Theorem] again, we get

$$S^\cdot = \sum Rj_{V*}IC^\cdot(\overline{V}; L_V)[-\operatorname{reldim}(j_V)] \quad \text{(up to cobordism)}$$

where $\{V\}$ is a stratification of X and L_V is a certain local system of finite-dimensional $\mathbf{Q}$-vector space on V. Note that each $IC^\cdot(\overline{V}; L_V)$ is a self-dual complex of sheaves on $\overline{V}$ (see [CS1, §2]) and that for the top-dimensional stratum V (i.e., $\overline{V} = X$)

$$\begin{aligned} Rj_{V*}IC^\cdot(\overline{V}; L_V)[-\operatorname{reldim}(j_V)] &= IC^\cdot(\overline{V}; L_V) \\ &= IC^\cdot(X; \mathbf{Q}^{n_X}), \quad \text{where } n_X = \operatorname{rank} H^0(y; S^\cdot(y)) \\ &= n_X IC^\cdot(X; \mathbf{Q}). \end{aligned}$$

Hence, we have (up to cobordism)

$$S^\cdot = n_X IC^\cdot(X; \mathbf{Q}) + \sum_{\dim V < \dim X} Rj_{V*}IC^\cdot(\overline{V}; L_V)[-\operatorname{reldim}(j_V)].$$

Then, since each $IC^\cdot(\overline{V}; L_V)$ is a self-dual complex of sheaves on $\overline{V}$, $\dim V < \dim X = k$, by the induction hypothesis, we can conclude the above lemma. □

Furthermore, by using resolution of singularities, we can prove the following lemma in a similar manner to that of the above lemma:

Lemma (2.4.2). *Given a self-dual complex $S^\cdot$ of sheaves of X, there exist finitely many smooth varieties W's, morphism $f_W: W \to X$, and integers n_W such that*

$$[S^\cdot] = \sum n_W f_{W\#}\mathbf{Q}_W[2\dim W].$$

Proof. Similarly as in the proof of the above lemma, we prove this lemma by induction on the dimension. If $S^\cdot$ is a self-dual complex of sheaves on X with $\dim X = 0$, then the statement of the above lemma is correct, so we suppose that the statement is true for $\dim X < k$ and prove the statement for $\dim X = k$. Let $S^\cdot$ be a self-dual complex of sheaves on X with $\dim X = k$. Then, by the above lemma, we have (up to cobordism)

$$S^\cdot = n_X IC^\cdot(X; \mathbf{Q}) + \sum_{\dim V < \dim X} Rj_{V*}IC^\cdot(\overline{V}; L_V)[-\operatorname{reldim}(j_V)].$$

Let $\pi: \widetilde{X} \to X$ be a resolution of singularities. Then we have (up to cobordism)

$$R\pi_* IC^\cdot(\widetilde{X}; \mathbf{Q}) = IC^\cdot(X; \mathbf{Q}) + \sum_{W \subset \operatorname{Sing}(X)} Rj_{W*}IC^\cdot(\overline{W}; L_W)[-\operatorname{reldim}(j_W)].$$

Hence

$$S^{\cdot} = n_X R\pi_* IC^{\cdot}(\widetilde{X}; \mathbf{Q}) + (-n_X) \sum_{W\subset \mathrm{Sing}(X)} Rj_{W^{\cdot}} IC^{\cdot}(\overline{W}; L_W)[-\mathrm{reldim}(j_W)]$$

$$+ \sum_{\dim V<\dim X} Rj_{V^{\cdot}} IC^{\cdot}(\overline{V}; L_V)[-\mathrm{reldim}(j_V)]$$

$$= n_X R\pi_* Q_{\widetilde{X}}[2\dim \widetilde{X}]$$

$$+ (-n_X) \sum_{W\subset \mathrm{Sing}(X)} Rj_{W^{\cdot}} IC^{\cdot}(\overline{W}; L_W)[-\mathrm{reldim}(j_W)]$$

$$+ \sum_{\dim V<\dim X} Rj_{V^{\cdot}} IC^{\cdot}(\overline{V}; L_V)[-\mathrm{reldim}(j_V)],$$

because $\widetilde{X}$ is smooth, so $IC^{\cdot}(\widetilde{X}; \mathbf{Q}) = \mathbf{Q}_{\widetilde{X}}[2\dim \widetilde{X}]$.

Then, since $IC^{\cdot}(\overline{V}; L_V)$ and $IC^{\cdot}(\overline{W}; L_W)$ are self-dual complex of sheaves on $\overline{V}$ and $\overline{W}$, $\dim V < \dim X = k$, and $\dim W < \dim X = k$, we can conclude the above lemma by the induction hypothesis. □

Now it is easy to see with this lemma that if $A_*, B_* : \Omega(\ ) \to H_*(\ ; \mathbf{Q})$ are two natural transformations satisfying the extra condition ("smooth L-condition") that if X is smooth

$$A_*([\mathbf{Q}_X[2\dim X]]) = B_*([\mathbf{Q}_X[2\dim X]]) = L^*(X) \cap [X],$$

then $A_* = B_*$. This completes the proof of the uniqueness. □

References

[BFM] P. Baum, W. Fulton, and R. MacPherson, *Riemann-Roch for singular varieties*, Inst. Hautes Étude Sci. Publ. Math. **45** (1975), 101–145.

[BBD] A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux pervers, analyse et topologie sur les espaces singulariers*, Astérisque **100** (1982), 1–171.

[Bo] A. Borel et al., *Intersection cohomology*, Progress in Math., vol. 50, Birkhäuser-Verlag, Boston, 1984.

[CS1] S. E. Cappell and J. L. Shaneson, *Stratifiable maps and topological invariants*, J. Amer. Math. Soc. **4** (1991), 521–551.

[CS2] ——, *Genera of algebraic varieties*, preprint, 1993.

[GM1] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology **19** (1980), 135–162.

[GM2] ——, *Intersection homology*. II, Invent. Math. **71** (1983), 77–129.

[Mac] R. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. (2) **100** (1974), 423–432.

Department of Mathematics, College of Liberal Arts, University of Kagoshima, Kagoshima 890, Japan

E-mail address: F77446@sinet.ad.jp